

# INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE 3RD DERIVATIVES BELONG TO $Q(I)$

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**ABSTRACT.** In this paper, we obtain some new inequalities of Hermite-Hadamard type and Simpson type for functions whose third derivatives belong to Godunova-Levin class.

## 1. INTRODUCTION

Following inequalities are well known in the literature as Hermite-Hadamard inequality and Simpson inequality respectively:

**Theorem 1.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . Then, the following double inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

**Theorem 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a four times continuously differentiable mapping on  $(a, b)$  and  $\|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$ . Then, the following inequality holds:*

$$\left| \frac{1}{3} \left[ \frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4.$$

In 1985, E. K. Godunova and V. I. Levin introduced the following class of functions (see [1]):

**Definition 1.** *A map  $f : I \rightarrow \mathbb{R}$  is said to belong to the class  $Q(I)$  if it is nonnegative and for all  $x, y \in I$  and  $\lambda \in (0, 1)$ , satisfies the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

In [2], Moslehian and Kian obtained Hermite-Hadamard and Ostrowski type inequalities for functions whose first derivatives belong to  $Q(I)$ .

To obtain our new results, it is necessary two lemmas .

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**Lemma 1.** [3] *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a three times differentiable function on  $I^\circ$  with  $a, b \in I$ ,  $a < b$ . If  $f''' \in L[a, b]$ , then*

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \\ &= \frac{(b-a)^3}{12} \int_0^1 t(1-t)(2t-1) f'''(ta + (1-t)b) dt. \end{aligned}$$

**Lemma 2.** [4] *Let  $f'' : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous function on  $I^\circ$  such that  $f''' \in L[a, b]$ , where  $a, b \in I$ ,  $a < b$ . Then*

$$\begin{aligned} & \int_a^b f(x) dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &= (b-a)^4 \int_0^1 p(t) f'''(ta + (1-t)b) dt, \end{aligned}$$

where

$$p(t) = \begin{cases} \frac{1}{6} t^2 (t - \frac{1}{2}), & \text{if } t \in [0, \frac{1}{2}] \\ \frac{1}{6} (t-1)^2 (t - \frac{1}{2}), & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

In this paper, using Lemma 1 and Lemma 2, we obtain some new inequalities for functions whose third derivatives belong to  $Q(I)$ .

## 2. MAIN RESULTS

We obtain the following new inequalities via Lemma 1.

**Theorem 3.** *Let  $f'' : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous function on  $I^\circ$  such that  $f''' \in L[a, b]$ , where  $a, b \in I$ ,  $a < b$ . If  $|f'''|^q$  belongs to  $Q(I)$ , then the following inequality holds*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{4} \right]^{\frac{1}{q}} \end{aligned}$$

for  $q \geq 1$ .

*Proof.* From Lemma 1 and using the power mean inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\ & \leq \frac{(b-a)^3}{12} \left( \int_0^1 t(1-t) |2t-1| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t(1-t) |2t-1| |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Then, since  $|f'''|^q$  belongs to  $Q(I)$ , we can write for  $t \in (0, 1)$

$$|f'''(ta + (1-t)b)|^q \leq \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t}.$$

Hence,

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left( \int_0^1 t(1-t) |2t-1| \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}} \\
& = \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left( \int_0^1 [(1-t) |2t-1| |f'''(a)|^q + t |2t-1| |f'''(b)|^q] dt \right)^{\frac{1}{q}} \\
& = \frac{(b-a)^3}{12} \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{4} \right]^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 t(1-t) |2t-1| dt = \frac{1}{16}$$

and

$$\int_0^1 t |2t-1| dt = \int_0^1 (1-t) |2t-1| dt = \frac{1}{4}.$$

The proof is completed.  $\square$

**Corollary 1.** *In Theorem 3, if we choose  $q = 1$  we obtain the following inequality*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{48} [|f'''(a)| + |f'''(b)|].
\end{aligned}$$

**Theorem 4.** *Let  $f'' : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous function on  $I^\circ$  such that  $f''' \in L[a, b]$ , where  $a, b \in I$ ,  $a < b$ . If  $|f'''|^q$  belongs to  $Q(I)$  and  $q > 1$ , then the following inequality holds*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{12} \frac{(\beta(q, q+1))^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} [|f'''(a)| + |f'''(b)|]^{\frac{1}{q}}
\end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $\beta(\cdot, \cdot)$  is Euler Beta function.

*Proof.* Since  $|f'''|^q$  belongs to  $Q(I)$ , from Lemma 1 and using the Hölder inequality we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{12} \left( \int_0^1 |2t-1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t^q (1-t)^q |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(b-a)^3}{12} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \int_0^1 t^q (1-t)^q \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}} \\
& = \frac{(b-a)^3}{12} \frac{(\beta(q, q+1))^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} [|f'''(a)|^q + |f'''(b)|^q]^{\frac{1}{q}},
\end{aligned}$$

which completes the proof.  $\square$

**Theorem 5.** *Under the assumptions of Theorem 4, we have the following inequality*

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{24} \left( \frac{1}{(p+1)(p+3)} \right)^{\frac{1}{p}} [|f'''(a)|^q + |f'''(b)|^q]^{\frac{1}{q}}
\end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Since  $|f'''|^q$  belongs to  $Q(I)$ , from Lemma 1 and using the Hölder inequality we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} [f'(b) - f'(a)] \right| \\
& \leq \frac{(b-a)^3}{12} \left( \int_0^1 t(1-t) |2t-1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t(1-t) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(b-a)^3}{12} \left( \int_0^1 t(1-t) |2t-1|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t(1-t) \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}} \\
& = \frac{(b-a)^3}{12} \left( \frac{1}{2(p+1)(p+3)} \right)^{\frac{1}{p}} \left[ \frac{|f'''(a)|^q + |f'''(b)|^q}{2} \right]^{\frac{1}{q}},
\end{aligned}$$

where we used

$$\int_0^1 t(1-t) |2t-1|^p dt = \frac{1}{2(p+1)(p+3)}.$$

The proof is completed.  $\square$

Following result is obtained via Lemma 2.

**Theorem 6.** *Let  $f'' : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous function on  $I^\circ$  such that  $f''' \in L[a, b]$ , where  $a, b \in I$ ,  $a < b$ . If  $|f'''|^q$  belongs to  $Q(I)$ , then the following*

inequality holds

$$\begin{aligned} & \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}} \left\{ \left( \frac{|f'''(a)|^q}{48} + \left( \frac{17}{48} - \frac{1}{2} \ln 2 \right) |f'''(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \left( \frac{17}{48} - \frac{1}{2} \ln 2 \right) |f'''(a)|^q + \frac{|f'''(b)|^q}{48} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

for  $q \geq 1$ .

*Proof.* Since  $|f'''|^q$  belongs to  $Q(I)$ , from Lemma 2 and using the power mean inequality we have

$$\begin{aligned} (2.1) \quad & \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^4}{6} \left\{ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) dt \right)^{1-\frac{1}{q}} \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left( \int_{\frac{1}{2}}^1 (t-1)^2 \left( t - \frac{1}{2} \right) dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 (t-1)^2 \left( t - \frac{1}{2} \right) |f'''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(b-a)^4}{6} \left( \frac{1}{192} \right)^{1-\frac{1}{q}} \left\{ \left( \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left( \int_{\frac{1}{2}}^1 (t-1)^2 \left( t - \frac{1}{2} \right) \left[ \frac{|f'''(a)|^q}{t} + \frac{|f'''(b)|^q}{1-t} \right] dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

If we use the inequalities below in (2.1), we get the desired result:

$$\int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) dt = \int_{\frac{1}{2}}^1 (t-1)^2 \left( t - \frac{1}{2} \right) dt = \frac{1}{192}$$

and

$$\begin{aligned} \int_0^{\frac{1}{2}} t^2 \left( \frac{1}{2} - t \right) \frac{1}{1-t} dt &= \int_{\frac{1}{2}}^1 \frac{1}{t} (t-1)^2 \left( t - \frac{1}{2} \right) dt \\ &= \frac{17}{48} - \frac{1}{2} \ln 2. \end{aligned}$$

□

**Corollary 2.** In Theorem 6, if we choose  $q = 1$  we obtain the following inequality

$$\begin{aligned} & \left| \int_a^b f(x)dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^4}{6} \left( \frac{3}{8} - \frac{1}{2} \ln 2 \right) [|f'''(a)| + |f'''(b)|]. \end{aligned}$$

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